QUANTUM HALL FLUIDS, LAUGHLIN WAVE FUNCTIONS, AND IDEALS IN THE WEYL ALGEBRA

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Abstract. It is known that noncommutative fluids used to model the Fractional Quantum Hall effect give Calogero–Moser systems. The group-theoretic description of these as reductions of free motion on type A Lie algebras leads directly to Laughlin wave functions. The Calogero–Moser models also parametrise the right ideals of the Weyl algebra, which can be regarded as labelling sources in the fluid.

1. Introduction

In this letter we draw attention to some connections between recent work on noncommutative fluids and the fractional quantum Hall effect, the Calogero–Sutherland and Calogero–Moser integrable models, and mathematical investigations of ideals in the Weyl algebra. The main result is a group theoretical derivation of Laughlin wave functions by considering the Calogero–Moser model as the reduction of free motion on the Lie algebra $su(N)$.

The transverse magnetic field B which generates the Hall effect forces the replacement of the translation group $E = \mathbf{R}^2$ of the plane sample by the magnetic translations, which form a projective representation of E with a multiplier determined by the field. Since translations in orthogonal directions no longer commute this suggests the use of the methods of noncommutative geometry, and Bellissard and Connes were able to interpret the conductance as a Chern character determined by pairing a cyclic 2-cocycle for the twisted group algebra of E determined by the multiplier with the projection onto the Fermi level and thence to obtain a very clear mathematical understanding of the integer Hall effect, [3,8].

To some extent this succeeded because the integer quantum Hall effect can be explained within one electron theory, whilst the fractional quantum Hall effect seems to be an essentially many body problem. Laughlin suggested early on that one should model the many electrons as an incompressible two-dimensional fluid, and there has been much work exploiting that idea [2,26]. Susskind suggested combining this approach with noncommutative geometry and looking at incompressible fluid flow in the noncommutative plane, [29], a suggestion soon taken up by others [27,17]. Douglas and Nekrasov have provided a useful review of noncommutative field theories in general [12].

The main result on Laughlin wave functions is derived in Section 4. It is preceded by brief accounts of quantum fluids and the Calogero type models in Sections 2 and 3. The final section describes the link between sources in the fluid and ideals in the Weyl algebra.

2. Quantum Hall fluids

Two dimensional incompressible fluids and the noncommutative plane share essentially the same symmetry group (at least as a discrete group, though not as an algebraic group, see [4]). To be more precise the noncommutative plane has a coordinate algebra generated by elements y^1 and y^2 which satisfy a commutation relation of the form

$$[y^j, y^k] = \theta^{jk} 1,$$

where θ^{jk} are the components of the symplectic form θ on E whose exponential defines a chosen multiplier on E. (We shall use natural identifications to identify θ as a non-degenerate antisymmetric bilinear form on the vector space E with a symplectic form on E considered as a differentiable manifold. To preserve flexibility we do not insist that this be the same as the multiplier for magnetic translations introduced above, though, since $\wedge^2 E$ is only one-dimensional, it will necessarily be a power of the magnetic multiplier.) We can then form the Weyl algebra of Schwartz functions on E with Moyal multiplication determined by θ. The automorphism group of the Weyl algebra consists of bijective maps from E to E which preserve the Schwartz functions, so that they should be differentiable (with some asymptotic conditions), and also preserve the Moyal product forcing them to preserve the symplectic form θ. The uniqueness of θ up to multiples, means that these diffeomorphisms are the same as those preserving the area two-form. On the other hand the flow of an incompressible fluid is described precisely by the volume-preserving (or in two dimensions, area preserving) diffeomorphisms of the region in which it is contained. (This description of fluid flow is clearly laid out in [1].)

One clear advantage of the noncommutative plane over the commutative one is that every automorphism (respectively, every derivation) is generalised inner, that is, can be implemented as conjugation by (respectively, commutator with) an element of the multiplier algebra. (For the algebra of Schwartz functions this just consists of tempered distributions whose products with any Schwartz functions are again Schwartz functions [Cor].) Some obvious derivations are actually inner. For example, (as noted already by Born, Jordan, and Dirac [5,11]) the map $f \mapsto [y^j, f]$ is easily checked to give $\theta^{jk} \partial f / \partial y^k$. So introducing the dual form θ_{jk}, we have

$$\partial f / \partial y^k = \theta_{kj}[y^j, f].$$

Any fluid motion adds to y^j a displacement term, and the commutator with the displaced fluid coordinate can be interpreted as a connection ∇ obtained by adding a function of the noncommutative coordinates to the partial derivative $\partial_k = \partial / \partial y^k$. More precisely, we write $\nabla = d + a$ for a connection for a finite rank projective module of the Weyl algebra (the analogue of a vector bundle over the noncommutative space), and this gives

$$[\nabla_k, f] = [\partial_k, f] + [a_k, f] = \theta_{kj}[y^j, f] + [a_k, f] = \theta_{kj}[y^j + \theta^{jl} a_l, f],$$

so that the connection replaces y^j by $z^j = y^j + \theta^{jl} a_l$. It is worth noting that the curvature of this connection is

$$[\nabla_j, \nabla_k] = \theta_{jr}\theta_{ks}[y^r + \theta^{rl} a_l, y^s + \theta^{sm} a_m] = \partial_j a_k - \partial_k a_j + [a_j, a_k] + \theta_{jk}.$$

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Due to the Moyal product, $[a_j, a_k]$ can contribute even for abelian gauge groups, and the last term arises from $[y^r, y^s]$.

The resulting reinterpretation of the fluid flow as a theory of connections has been used in the physics literature to recast the theory as a noncommutative Chern–Simons theory [29,27]. The gauge group is often said to be the whole group of unitary inner automorphisms, but this ignores the important asymptotic conditions. These have recently been discussed by Harvey [15], who argues that, rather than the unitary group of the Hilbert space, as is often assumed, the gauge group of the noncommutative theory should only consist of unitary functions with asymptotic values in the group of unitaries U such that $U - 1$ is asymptotically a compact operator. In any case the gauge group action allows us to gauge away any incompressible fluid motion (or at least any with the same asymptotic conditions). It cannot, however, remove sources or vortices which exist within the fluid. One is left with a physical picture in which the filled or partially filled Landau levels of the system are modelled as a fluid, but there are in addition some additional particles, quasi-particles, or sources. The gauge theory enables us to ignore the fluid and concentrate on these.

3. The Calogero–Moser model

Susskind started investigation of the resulting noncommutative Chern–Simons theory by looking at matrix versions. Although there are no finite-dimensional solutions to the commutation relation $[y^j, y^k] = \theta^{jk}1$, an heuristic argument suggests that if the fluid contains a source of strength q in a state ψ then the commutation relation is amended to $[y^j, y^k] = \theta^{jk}(1 - q|\psi\rangle\langle\psi|)$, which (taking traces) has $N \times N$ matrix solutions if and only if $q\|\psi\|^2 = N$. Polychronakos was able to reach a similar conclusion in a more direct way by consideration of a theory containing additional source or droplet fields transforming with the fundamental representation of $U(N)$.

It is known that there are solutions of this modified commutation relation and that they describe solutions of the Calogero–Moser model [25,19]. We briefly outline the latter approach. This starts by considering the cotangent bundle $T^*u(N) \sim u(N) \oplus u(N)^*$ of the Lie algebra $u(N)$ of the unitary group $U(N)$. We identify $u(N)$ and $u(N)^*$ with the self-adjoint $N \times N$ matrices, and write (X, P) for a typical element of $T^*u(N) \sim u(N) \oplus u(N)^*$. The dual $u(N)^*$ can also be identified with $u(N)$ using the trace (Hilbert–Schmidt) inner product. The unitary group acts by conjugation. This is a symplectic map (with respect to the standard symplectic structure on the cotangent bundle $T^*u(N)$, and we readily calculate that the moment map $\mu : T^*u(N) \to u(N)^* \cong u(N)$ is given by $\mu(X, P) = [P, X]$. Identifying P and X with y^1 and y^2, the unitary equivalence class of matrix solutions of the modified commutation relation can therefore be found by symplectic reduction to $\mu^{-1}(\theta^{12}(1 - q|\psi\rangle\langle\psi|)/U_\psi(N)$, where $U_\psi(N)$ is the stabiliser (little group) of the projection onto ψ.

On the other hand each unitary orbit can be parametrised as follows. The eigenvalues $\{x_1, x_2, \ldots, x_N\}$ of X are unitary invariants. Let $\{e_1, e_2, \ldots, e_N\}$ be the corresponding eigenvectors, and $p_j = \langle e_j|Pe_j\rangle$ the diagonal entries of P. The commutation relation tells us that if $\psi = q^{-\frac{1}{2}} \sum e_j$ (which is consistent with the normalisation condition) then the x_j must be distinct and that, the off-diagonal entries ($j \neq k$) of P are given by $\langle e_j|Pe_k\rangle =$

$\theta^{jk}(x_j - x_k)^{-1}$. This enables us to parametrise the orbit by $(x_1, \ldots, x_N, p_1, \ldots, p_N)$. These coordinates are unique up to permutations of the N indices, and this is the only vestige of the original unitary symmetry which survives the reduction. The Hamiltonian $\frac{1}{2}\mathrm{tr}(P^2)$ in the unreduced space reduces to the Calogero–Moser Hamiltonian

$$\frac{1}{2}\sum_{j=1}^{N} p_j^2 - \frac{1}{2}\sum_{j<k} \theta_{jk}^2 (x_j - x_k)^{-2}.$$

(A similar analysis on $T^*U(N)$ gives the Calogero–Sutherland model for particles on the circle instead of the real line. This case is particularly interesting because a rigorous second quantisation has recently been described by Carey and Langmann [7].)

It is a standard mathematical trick that such a reduction can also be achieved by reduction of the larger space $T^*u(N) \times \mathbf{C}^N$. The second term is equipped with the imaginary part of the inner product as symplectic form. The moment map for the conjugation action on $T^*u(N)$ combined with the natural action of $U(N)$ on $V = \mathbf{C}^N$ is $\mu(X, P, \psi) = [X, P] - |\psi\rangle\langle\psi|$, and reduction at $\theta^{12}1$, gives us the same reduced manifold as before. (The stabiliser is now $U(N)$ whose action is enough to remove the extra degrees of freedom which we had introduced.) This construction actually recovers the extra fields which Polychronakos introduced for physical reasons [26,27]. (Although introduced as fields, the constraint equations force determine them up to a vector in V.)

An even more subtle variant appears in [30], where everything is complexified, and one considers the $GL(N)$ action on $T^*(gl(N) \oplus V)$. The moment map on an element $(X, P, v, w) \in T^*(gl(N) \oplus V) \cong gl(N) \oplus gl(N)^* \oplus V \oplus V^*$ is

$$\mu(X, P, v, w) = [X, P] - |v\rangle\langle w|,$$

and reduction gives the complex Calogero–Moser equation. (As Wilson notes the complex version allows collisions between particles, but the unreduced space desingularises the effect of these collisions [30].)

4. The quantised Calogero–Sutherland and Calogero–Moser models

The quantised version of reduction is to exponentiate the point of the dual Lie algebra at which the system is reduced, to a (linear) character of the stabiliser, and then to restrict attention to the Hilbert subspace on which it acts by this character. We work with the second version of the construction on $T^*u(N) \times V$. In fact, since the adjoint action on $u(N)$ is trivial on the orthogonal complement of $su(N)$ we restrict to that It is thus convenient to use the Schrödinger Hilbert space $L^2(su(N))$ for the first factor and the Fock space $\bigoplus_{r=1}^{\infty} \otimes_S^r V$ for the second factor. We first remove the factor in front of the vector part (to agree with the unit factor in the moment map) by rescaling to $X = q^{-\frac{1}{2}}y^1$, $P = q^{-\frac{1}{2}}y^2$ and then reduce at θ^{12}/q. Analysis of this reduction goes back to [27,17], but we shall analyse it within the standard setting of compact group representation theory, leaving physical considerations till the end.

PROPOSITION. The exponential of $q^{-1}\theta^{12}$ defines a character of $U(N)$ only when $\theta^{12} = kq$ for integral k. Thus the filling factor $\nu = q/\theta^{12} = 1/k$ is the reciprocal of

an integer. The corresponding reduced space describes fermionic or bosonic particles for odd or even k respectively. For a given value of k the reduced space involves only the component $\oplus_S^{kN}V$ in the Fock space of V.

Proof. Any character of $U(N)$ must be lifted from its quotient by the commutator subgroup $SU(N)$, and it is easy to check that they are just (integral) powers of the determinant: $U \mapsto \det(U)^k$. Exponentiating θ^{12}/q tells us that the relevant character of $U(N)$ is given by $k = \theta^{12}/q$, and shows, in particular, that $q^{-1}\theta^{12} = k$ is an integer. As we noted earlier the permutation group is the vestige of the unitary gauge group which survives reduction. A permutation matrix has determinant 1 or -1 as the permutation is odd or even, so the quantised particles are bosons or fermions according to whether the integer is even or odd. Physically $\nu = q/\theta^{12}$ is interpreted as the filling factor, so we have shown that it is the reciprocal of an integer, and for fermionic behaviour it must be an odd integer.

To find the isotypic component transforming with this character, we first consider just the scalars $\lambda 1 \in U(N)$. These have a trivial conjugation action on $su(N)$, and so the representation on $L^2(su(N))$ is trivial. On the other hand the action on $\otimes_S^r V$ is by λ^r. Noting that $\det(\lambda 1)^k = \lambda^{Nk}$, we see that $r = Nk = N/\nu$ is permitted. $\quad\square$

This result holds equally well for the Calogero–Sutherland model where the space $L^2(su(N))$ is replaced by $L^2(SU(N))$, whilst the mathematics becomes particularly transparent in that case so we shall start with that. We shall show, by considering the action of $SU(N)$, that for integer values of k there is indeed a non-trivial reduced space, which can be described very precisely in terms of Laughlin type wave functions. The principle of the argument is already apparent for $k = 1$ so we first deal with that.

THEOREM. The part of $L^2(SU(N)) \otimes \otimes_S^N V$ which transforms with $\det(U)$ under the action of $U \in U(N)$ has a distinguished cyclic vector Δ_0 (considered as a $\otimes_S^N V$-valued function of $Z \in SU(N)$) defined by

$$\langle v^{(N)}|\Delta_0(Z)\rangle = \langle Z^{*N-1}v \wedge \ldots \wedge Z^{*2}v \wedge Z^*v \wedge v|\epsilon\rangle,$$

where $v^{(N)} = v \otimes v \otimes \ldots \otimes v \in \otimes_S^N V$, and ϵ is a fixed unit vector in $\bigwedge^N V$, and it is spanned by functions of the form $\Delta_{\mathbf{k}}(Z) = \chi_{\mathbf{k}}(Z)\Delta_0(Z)$, where $\chi_{\mathbf{k}}$ is the character of the irreducible representation of $SU(N)$ with highest weight $\mathbf{k}$.

Proof. We have already dealt with the action of multiples of the identity. (Moreover, every unitary matrix can be written as λU with $U \in SU(N)$ and $\lambda \in \mathbf{C}$, unique up to multiplying λ by an N-th root of unity, and in $\otimes_S^N V$ this ambiguity has no effect.)

By the Peter–Weyl theorem $L^2(SU(N))$ decomposes under the left and right actions of $SU(N) \times SU(N)$ as the direct sum of terms $D \otimes D^*$ where D ranges over the irreducible representations of $SU(N)$. We are interested only in the adjoint action of the diagonal subgroup $SU(N) \subset SU(N) \times SU(N)$ and seek those D for which $D \otimes D^* \otimes \otimes_S^N V$ carries a trivial $SU(N)$ representation. This means that we need a non-trivial intertwining operator T from D to $D \otimes \otimes_S^N V$. The Littlewood–Richardson rule [22,23] tells us that this is possible if and only if D has highest weight $\mathbf{k} = (k_1, k_2, \ldots, k_{N-1})$ with $k_1 > k_2 > \ldots > k_{N-1} > 0$. Moreover, when this constraint is satisfied there is, up to multiples, a unique intertwining

operator. The minimum highest weight permitted by this constraint has $k_j = N - j$, so that $\mathbf{k} = (N - 1, N - 2, \ldots, 1) = \rho$ is precisely half the sum of the positive roots.

The corresponding representation D_ρ on a space $\mathcal{H}_\rho$ can also be realised as the highest weight irreducible summand in the tensor product representation D_Λ on $\Lambda = V \otimes \wedge^2 V \otimes \ldots \otimes \wedge^{N-1} V$ given by the natural action on V. It is easy to write down an intertwining operator from Λ to $\Lambda \otimes \otimes_S^N V$. For $x = x_1 \otimes x_2 \otimes \ldots \otimes x_{N-1}$ and $y = y_1 \otimes y_2 \otimes \ldots \otimes y_{N-1}$, with $x_j, y_j \in \wedge^j V$, $\epsilon \in \wedge^N V$, and $v \in V$ we set

$$\langle y \otimes v^{(N)} | T(x) \rangle = \langle v | x_1 \rangle \langle v \wedge y_1 | x_2 \rangle \ldots \langle v \wedge y_{N-2}) | x_{N-1} \rangle \langle v \wedge y_{N-1} | \epsilon \rangle.$$

The Peter–Weyl Theorem associates to the operator T_v defined by

$$\langle y | T_v(x) \rangle = \langle y \otimes v^{(N)} | T(x) \rangle,$$

the function on G given by $Z \mapsto \operatorname{tr}(T_v D_\rho(Z)^*)$. Another application of the Littlewood–Richardson rule shows that the only contribution to the trace comes from the highest weight components, so we may as well replace D_ρ by D_Λ, which is much easier to compute as the sum of terms $\langle x | T_v(x) \rangle$ with x in an orthonormal basis, that is each x_j in an orthonormal basis of $\wedge^j V$. The definitions give

$$\langle x | T_v D_\Lambda(Z)^* x \rangle = \langle x \otimes v^{(N)} | T D_\Lambda(Z)^* x \rangle$$
$$= \langle v \wedge x_{N-1} | \epsilon \rangle \langle v | Z^* x_1 \rangle \langle v \wedge x_1 | \wedge^2 Z^* x_2 \rangle \ldots \langle v \wedge x_{N-2} | \wedge^{N-1} Z^* x_{N-1} \rangle.$$

Summing over an orthonormal basis of x_1 we find that

$$\sum_{x_1} \langle v | Z^* x_1 \rangle \langle v \wedge x_1 | \wedge^2 Z^* x_2 \rangle = \sum_{x_1} \langle Zv | x_1 \rangle \langle Zv \wedge Zx_1 | x_2 \rangle = \langle Zv \wedge Z^2 v | x_2 \rangle.$$

Multiplying this by $\langle Zv \wedge \wedge^2 Zx_2 | x_3 \rangle$ and summing over orthonormal x_2 gives

$$\sum_{x_2} \langle Zv \wedge Z^2 v | x_2 \rangle \langle Zv \wedge \wedge^2 Zx_2 | x_3 \rangle = \langle Zv \wedge Z^2 v \wedge Z^3 v | x_3 \rangle,$$

and inductively the sequence collapses down to $\langle Zv \wedge Z^2 v \wedge \ldots \wedge Z^N v | \epsilon \rangle$. Since ϵ is invariant under $Z \in SU(N)$ this can also be written as $\langle v \wedge Zv \wedge \ldots \wedge Z^{N-1} v | \epsilon \rangle$, or as $\langle Z^{-(N-1)} v \wedge \ldots \wedge Z^{-1} v \wedge v | \epsilon \rangle$. This last version extends from $SU(N)$ to $U(N)$, since the action of $U \in U(N)$ gives

$$\langle v^{(N)} | U^{(N)} \Delta_0(U^{-1} Z U) \rangle = \langle (U^{-1} v)^{(N)} | \Delta_0(U^{-1} Z U) \rangle$$
$$= \langle (U^{-1} Z U)^{-(N-1)} U^{-1} v \wedge \ldots \wedge U^{-1} v | \epsilon \rangle$$
$$= \langle U^{-1} Z^{-(N-1)} v \wedge \ldots \wedge U^{-1} v | \epsilon \rangle$$
$$= \langle Z^{-(N-1)} v \wedge \ldots \wedge v | \wedge^N U \epsilon \rangle$$
$$= \det(U) \langle (v^{(N)} | \Delta_0(Z) \rangle,$$

showing that Δ_0 is simply multiplied by $\det(U)$ and so in the required subspace. Replacing $\Delta_0(Z)$ by its k-th symmetric tensor power gives similarly a $\otimes_S^{Nk} V$ valued function which transforms with $\det(U)^k$.

Almost all unitary matrices Z have eigenvectors $e_1, \ldots, e_N$ with distinct eigenvalues of modulus 1 $z_1, \ldots, z_N$ we may take $\epsilon = e_1 \wedge \ldots \wedge e_N$, and then

$$
\begin{aligned}
\langle v^{(N)} | \Delta_0(Z) \rangle &= \langle Z^{-(N-1)} v \wedge \ldots \wedge v | \epsilon \rangle \\
&= \det(\langle v | Z^{N-s} e_r \rangle) \\
&= \det(z_r^{N-s} \langle v | e_r \rangle) \\
&= \Delta(\mathbf{z}) \prod_{r=1}^{N} \langle v | e_r \rangle,
\end{aligned}
$$

where $\Delta(\mathbf{z}) = \prod_{r<s}(z_r - z_s)$. Thus $\Delta_0(A) = \Delta(\mathbf{z}) \widetilde{e}$, where we have written $\widetilde{e}$ for the symmetric tensor product of the eigenvectors of Z. Although Z has distinct eigenvalues on all but a closed submanifold of lower dimension, there remains some subtlety when A passes through such a submanifold, since the eigenvectors can then be permuted. However, conjugation invariance means that this problem can be avoided by selecting a Cartan subgroup and arranging the ordering in advance. This shows that, unlike the scalar function Δ, the vector-valued function Δ_0 can be extended from a Cartan subgroup to the whole group.

Now the general function in $L^2(SU(N)) \otimes \otimes_S^N V$ can be written in the form $\chi(Z) \Delta_0(Z)$, where $\chi(U^{-1} Z U) = \chi(Z)$ in order that it transform with $\det(U)$, that is χ is a central function. The central functions are spanned by the irreducible characters $\chi_{\mathbf{k}}$, so that the functions $\Delta_{\mathbf{k}} = \chi_{\mathbf{k}} \Delta_0$ span the space transforming with $\det(U)$. Using Weyl's character formula and earlier notation, we could write $\chi_{\mathbf{k}}(Z) = \det(z_r^{k_s+N-s})/\Delta(\mathbf{z})$, so that

$$
\langle v^{(N)} | \Delta_{\mathbf{k}}(Z) \rangle = \langle Z^{-k_1-(N-1)} v \wedge Z^{-k_2-(N-2)} v \wedge \ldots \wedge v | \epsilon \rangle \widetilde{e}.
$$

The inner product on the reduced space involves only integration over a cross-section of adjoint orbits. It is therefore sufficient to integrate over a Cartan subgroup H of diagonal matrices. Thus

$$
\langle \Delta_{\mathbf{k}} | \Delta_{\mathbf{l}} \rangle = \int_H \langle \Delta_{\mathbf{k}}(h) | \Delta_{\mathbf{l}}(h) \rangle \, dh,
$$

where the second inner product is that in $\otimes_S^N V$. Substituting the formula for $\Delta_{\mathbf{k}}$ gives

$$
\langle \Delta_{\mathbf{k}} | \Delta_{\mathbf{l}} \rangle = \int_H \overline{\chi}_{\mathbf{k}}(h) \chi_{\mathbf{l}}(h) |\Delta(h)|^2 \, dh = \int_{SU(N)} \overline{\chi}_{\mathbf{k}}(Z) \chi_{\mathbf{l}}(Z) \, dZ,
$$

the normal inner product on $L^2(SU(N))$. $\quad \square$

This result has an easy extension to the Calogero–Moser model.

COROLLARY. The part of $L^2(su(N)) \otimes \otimes_S^N V$ which transforms with $\det(U)$ under the action of $U \in U(N)$ has a distinguished cyclic vector δ_0 (considered as a $\otimes_S^N V$-valued function of Z in $su(N)$) (identified with self-adjoint matrices) defined by

$$\langle v^{(N)} | \delta_0(Z) \rangle = \langle Z^{N-1} v \wedge \ldots \wedge Z^2 v \wedge Z v \wedge v | \epsilon \rangle,$$

where $v^{(N)} = v \otimes v \otimes \ldots \otimes v \in \otimes_S^N V$, and ϵ is a fixed unit vector in $\bigwedge^N V$, and it is spanned by functions of the form $\delta_{\mathbf{k}}(Z) = \chi_{\mathbf{k}}(Z) \delta_0(Z)$, where $\chi_{\mathbf{k}}$ is the extension to general matrices of the polynomial giving the character of the irreducible representation of $SU(N)$ with highest weight $\mathbf{k}$.

Proof. There are two ways to derive this result. The direct method is to note that $T^* u(N)$ can be identified with the complexified Lie algebra $u(N)_{\mathbf{C}}$, on which there are mutually commuting actions of $U(N)$ by left and right multiplication. Howe duality [16] on the quantised space $L^2(u(N))$ shows that it decomposes exactly as in the Peter–Weyl theorem. Alternatively, one may use the Cayley transform

$$Z \mapsto U_Z = (1 - iZ)(1 + iZ)^{-1} [\det(1 + iZ) / \det(1 - iZ)]^{-1/N} \in SU(N),$$

valid on all but a null set, to identify $L^2(su(N))$ with $L^2(SU(N))$. (As noted before the ambiguity in the root of the determinant has no effect in this case, and the map commutes with the adjoint actions of $SU(N)$ on itself and its Lie algebra.) Then we simply apply the previous arguments to obtain a distinguished wave function $\delta_0' \in L^2(su(N)) \otimes \otimes_S^N V$ such that, for $Z \in su(N)$

$$\langle v^{(N)} | \delta_0'(Z) \rangle = \langle U_Z^{-(N-1)} v \wedge \ldots \wedge U_Z^{-2} v \wedge U_Z^{-1} v \wedge v | \epsilon \rangle,$$

Replacing v by $(1 + iZ)v$ and recalling that $\wedge^N (1 + iZ) = \det(1 + iZ)$ we obtain

$$\det(1 - iZ)^{(N-1)} \langle v^{(N)} | \delta_0'(Z) \rangle$$
$$= \langle (1 - iZ)^{-(N-1)} v \wedge (1 - iZ)^{-(N-2)} (1 + iZ) v \wedge \ldots \wedge (1 + iZ)^{N-1} v | \epsilon \rangle,$$

which reduces to a numerical multiple of $\langle Z^{N-1} v \wedge Z^{N-2} v \wedge \ldots \wedge v | \epsilon \rangle$, essentially the same function as for the Calogero–Sutherland model. The rest of the result follows as before, and the isotypic subspace is spanned by vectors

$$\chi_{\mathbf{k}}(\mathbf{z}) \delta'(Z) = \det(z_r^{k_s + N - s}) \widetilde{e}.$$

The Cayley transform has Radon–Nikodym derivative $|\det(1 + iZ)|^{-2}$, so giving a Cauchy measure on $su(N)$. However, because we used only the adjoint action of $SU(N)$ on $L^2(SU(N))$ rather than the action of $SU(N) \times SU(N)$ one may change the measure on the Cartan subalgebra h which provides a cross-section of the orbits. $\square$

In the physics literature it is customary to choose a Gaussian measure $\exp[-\frac{1}{2} \omega \operatorname{tr}(Z^2)]$ appropriate to changing the free motion on $su(N)$ to an oscillator motion with Hamiltonian $\frac{1}{2} \operatorname{tr}(P^2 + \omega^2 X^2)$. (Polychronakos motivates this normalisation factor by adding an

appropriate harmonic oscillator potential to the action.) Since $\text{tr}(Z^2)$ is invariant under the adjoint action the transformation properties are not compromised. In fact for such oscillatory motion it is more natural to do a Bargmann transform from $L^2(SU(N))$ to the space of holomorphic functions of $Z \in su(N)_{\mathbf{C}}$ which are square-integrable with respect to the measure $\exp[-\frac{1}{2}\omega\text{tr}(Z^*Z)]$. Alternatively one multiplies the wave functions by $\exp[-\frac{1}{4}\omega\text{tr}(Z^*Z)]$ rather than by $\det(1-iZ)^{-(N-1)}$, giving precisely the Laughlin wave function (of non-self-adjoint Z)

$$\langle v^{(N)}|\delta_0(Z)\rangle = \langle Z^{*N-1}v \wedge Z^{*N-2}v \wedge \ldots \wedge v|\epsilon\rangle \exp[-\frac{1}{4}\omega\text{tr}(Z^*Z)].$$

Our functions Δ_0 and δ_0, derived by these group-theoretic arguments, are essentially the same as the ground state which appears in [17], but our factorisation makes clearer the analogy with the Laughlin ground state. They also appear in correlation functions for the second quantised Calogero–Sutherland model [7]. It should be noted however, that the second quantised Calogero–Sutherland model is not simply a quantisation of this model, since it combines all particle numbers and so all values of N together.

5. Sources and ideals in the Weyl algebra

A very different approach to the theory can be made by returning to the idea of sources in the fluid. The interpretation of diffeomorphisms as a gauge theory meant that we could effectively remove regular incompressible fluid motions by a gauge transformation. The connection can, however, still encode for sources. In the commutative theory of incompressible flow these N sources would have positions, which could be defined by an ideal of index N in the coordinate ring $\mathbf{C}[x,y]$ which vanishes at precisely those N points. (In the spirit of algebraic geometry we work now in the complex plane.)

The obvious generalisation to noncommutative fluids would be to study ideals in the noncommutative ring $\mathcal{A}_1 = \mathbf{C}[y^1, y^2]$, where $[y^1, y^2] = \theta 1$. We need now to be more precise and work with right ideals, $\mathcal{I}$, but clearly $\mathcal{I}$ cannot have finite index as the quotient $\mathcal{A}_1/\mathcal{I}$ would then provide a finite dimensional representation of the commutation relations. However, Cannings and Holland have classified the right ideals in $\mathcal{A}_1$, by constructing an isomorphism to a Grassmannian of subspaces of the rational functions $\mathbf{C}(z)$, [6]. Berest and Wilson, [4], note that the same Grassmannian arises in Wilson's work on the link between the Calogero–Moser model and the KP hierarchy [30], where it is shown to be isomorphic to the disjoint union of spaces

$$\mathcal{C}_N = \{(X, Y) \in gl(N, \mathbf{C}) : \text{rk}([X, Y] - 1) \leq 1\},$$

(where rk denotes the rank of a matrix). (Berest and Wilson note their later discovery that the unpublished thesis of Kouakou [20] had anticipated parametrisation by an integer N.) The unitary equivalence classes of the pairs (X, Y) give precisely the reduction studied earlier and, as already noted, can be identified with the (complex) N-particle Calogero–Moser system. Each right ideal $\mathcal{I}$ is associated under the isomorphisms with a point in one of the $\mathcal{C}_N$, so that although $\mathcal{I}$ has infinite index, it still defines a system of N points, and, moreover, these have a natural Calogero–Moser dynamics.

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One of the major theorems of Berest and Wilson, says that the natural action of the automorphism group $\mathrm{Aut}(\mathcal{A}_1)$ on the right ideals pulls back to a transitive action on each individual $\mathcal{C}_N$. As they observe, this is surprising because it is much stronger than the corresponding result in the commutative case, where the action cannot be transitive since it cannot move between configurations where the points are distinct and those in which some points are coincident. As often happens quantisation desingularises.

Finally picking up on another observation of Berest and Wilson, we note that the space which reduced to the Calogero–Moser model has an obvious generalisation to $T^*(gl(N) \oplus M_{N,r})$, where $M_{N,r}$ denotes the $N \times r$ matrices (over $\mathbf{C}$). Retaining the earlier notation but with $v \in M_{N,r}$ and $w \in M_{r,n}$, the corresponding moment map is given by $\mu(x,p,v,w) = [X,Y] - vw^*$. However, the map $j : (X,Y,v,w) \mapsto (Y^*, -X^*, w^*, -v^*)$ which anticommutes with i and has square -1, provides a hyperkähler structure from which one can obtain an additional (real) moment map,

$$\mu_{\mathbf{R}}(X,Y,v,w) = \frac{1}{2}\left([X,X^*] + [Y,Y^*] + vv^* - w^*w\right).$$

The hyperkähler reduction agrees with the previous reduction and gives the same space $\mathcal{C}_N(r)$:
$$\mu^{-1}(1) \cap \mu_{\mathbf{R}}^{-1}(0)/U(N) = \mu^{-1}(1)/GL(N,\mathbf{C}).$$
Multiplication by $(i+j)/\sqrt{2}$ gives an isomorphism with another reduction

$$\mathcal{D}_N(r) = \mu^{-1}(0) \cap \mu_{\mathbf{R}}^{-1}(1)/U(N),$$

which describes the noncommutative ADHM data for the construction of an r-instanton solutions of noncommutative self-dual Yang–Mills theory, [24]. This links the subject with other recent work on noncommutative instantons, [9,28,18,12,14], and shows that the Calogero–Moser model can be regarded as a one instanton solution. This also opens the way to the description of more general filling fractions of the form r/k.

There remains the question of whether the noncommutative incompressible fluid model can be derived directly from quantum field theory, and here we note that Fröhlich and Studer found a limiting commutative Chern–Simons action for non-relativistic charged spinning electrons [13], and recently Lieb, Seiringer and Yngvason have derived the Gross–Pitaevskii energy functional directly from a bosonic field theory [21], providing hope that this can be done. (It is interesting that in the Lieb–Seiringer–Yngvason limit the part of the two body interaction potential which survives is precisely the inverse square part as appears in the Calogero–Moser model.)

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